TABLE 2

α, deg	HQ	IQ	GQ	UHM
30	7,4	0,33	0,12	0,45
60	4,0	0,39	0,17	0,53
90	2,5	0,47	0,25	0,60
120	1,5	0,57	0,35	0,66
150	0,66	0,70	0,51	0,67

TABLE 3

α, deg	$H\cdot 10^3$, m	U, m/sec	$H\cdot 10^4$, m	U, m/sec
	Water		Nickel	
30	4,2	1,25	21	0,59
60	2,3	1,7	11	0,69
90	1,4	1,9	6,9	0,79
120	0,84	2,1	4,1	0,86
150	0,37	2,1	1,8	0,86

As an example we calculate the flow parameters for rivulets of water over a vertical wall and rivulets of molten nickel on a rapidly rotating horizontal disk. For water $\rho = 10^3$ kg/m^3, $\nu = 10^{-6}$ m^2/sec, $a = 9.8$ m/sec^2, $Q = 10^{-6}$ m^3/sec, and for nickel at 1800 K, $\nu = 6.4\cdot 10^{-7}$ m^2/sec, disk radius is 10^{-1} m, and the number of revolutions is 104.7 rad/sec, $Q = 10^{-9}$ m^3/sec (Table 3). The radio of Coriolis to centrifugal acceleration is estimated from

$$\frac{2\omega v}{\omega^2 r} = \frac{2v}{\omega r} = 2\left(\frac{Qa}{\nu}\right)^{1/2}\frac{UQ(\alpha)}{\omega r} = 2\left(\frac{Q}{\nu r}\right)^{1/2} UQ(\alpha), \quad a = \omega^2 r.$$

The maximum of ratio $2v/\omega r = 0.16$ is achieved with $\alpha = 150°$. In [1, 2, 5, 6] the two-dimensional velocity field is found in the form $v = v[y/\delta(x)]$, where the parabolic profile is taken from the solution for the unidimensional problem for a film. With this choice of velocity field the boundary condition at the free rivulet surface is not observed.

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PERTURBATION METHOD COMPUTATION OF THE MAXIMAL GROUP VELOCITIES OF INTERNAL WAVES IN A STRATIFIED MEDIUM WITH MEAN SHEAR FLOWS

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UDC 551.466

Propagation of internal gravitational waves excited in a stratified fluid layer $-H < z < 0$, $-\infty < x, y < \infty$ with mean horizontal shear flows is described by the equation [1]

$$Lu(t, x, y, z, z_0) = Q(t, x, y, z, z_0), \; u = 0(z = 0, -H), \tag{1}$$

where the operator is

$$L = \frac{D^2}{Dt^2}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] - \frac{D}{Dt}\left[U''_{zz}\frac{\partial}{\partial x} + V''_{zz}\frac{\partial}{\partial y}\right] + N^2\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right];$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y};$$

$U = U(z)$, $V = V(z)$ are the velocity components of the mean flow $\mathbf{U} = \{U, V, 0\}$ at the horizon z, and $N(z)$ is the Brunt–Väisälä frequency. The Boussinesq and solid covers are used. The Miles stability condition $Ri(z) = N^2(z)/[(U'_z)^2 + (V'_z)^2] > 1/4$ is assumed satisfied and

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assuring that the internal waves cannot be exchanged with mean shear energy flows.

The Green's function of the boundary value problem (1) is found in [2], i.e., its solution $\Gamma(t, x, y, z, z_0)$ for $Q = \delta(t)\delta(x)\delta(y)\delta(z - z_0)$ which vanishes identically for $t < 0$ and the asymptotic of Γ is determined as $t \to \infty$ and for fixed $x/t = U_x$, $y/t = U_y$. A Fourier transform in the variables t, x, y is used to find Γ, which reduces the problem to determining the vertical Green's function $G(\omega, \lambda, \mu, z, z_0)$: $L_0\left(\omega, \lambda, \mu, z, \frac{\partial}{\partial z}\right)G = (\omega - f)^2\,\delta(z - z_0)$, $G = 0$ $(z = 0, -H)$. Here L_0 is the Taylor–Goldstein operator $L_0 u = (\omega - f)^2 u''_{zz} + \left[k^2(N^2 - (\omega - f)^2) + f''_{zz}(\omega - f)\right]u$; $f = f(z) = \lambda U(z) + \mu V(z)$; $k^2 = \lambda^2 + \mu^2$. The function G as a function of ω has simple poles for real $\omega = \omega_n(\lambda, \mu)$ which are the eigennumbers of the boundary value problem

$$L_0\left(\omega_n, \lambda, \mu, z, \frac{\partial}{\partial z}\right)\varphi_n = 0, \quad \varphi_n = 0\,(z = 0, -H), \tag{2}$$

and a slit on the real ω axis connecting the branch points $\omega = \min_z f$ and $\omega = \max_z f$.

The passage to the function Γ by means of the inverse Fourier transform yields the expression

$$\Gamma = \sum_n \Gamma_n + \Gamma_H, \tag{3}$$

where Γ_n corresponds to the contribution from the n-th pole $\omega = \omega_n(\lambda, \mu)$ of the function G and Γ_H is the integral of the function G along the slit. The function Γ_H turns out to be negligibly small for $t \gg 1$ compared to Γ_n of the form

$$\Gamma_n = \operatorname{Im}\int\int_{-\infty}^{\infty} A_n(\lambda, \mu, z, z_0)\exp i\left[\lambda x + \mu y - \omega_n(\lambda, \mu)t\right]d\lambda d\mu. \tag{4}$$

The expression for A_n is presented in [2] (it is not required later), and ω_n is the eigennumber of the boundary value problem (2). Summation over all $\omega_n(\lambda, \mu) < \min_z f(z)$ is taken in (3); these eigennumbers are enumerated in increasing order.

To find the asymptotic Γ_n for t, $|x|$, $|y| \gg 1$ we set $x = \alpha t$, $y = \beta t$, $t \gg 1$, i.e., we seek the asymptotic Γ_n for $t \gg 1$ at the observation point x, y receding from the origin at the velocity $U = (\alpha, \beta)$. Then the phase function in (4) is written in the form $S = t(\alpha\lambda + \beta\mu - \omega_n(\lambda, \mu))$ and its stationary points are determined from the equations

$$\alpha = \frac{x}{t} = \frac{\partial\omega_n}{\partial\lambda}, \quad \beta = \frac{y}{t} = \frac{\partial\omega_n}{\partial\mu}. \tag{5}$$

The set of points $\alpha = x/t$, $\beta = y/t$ in the plane α, β (i.e., the domain $\alpha = x/t$, $\beta = y/t$ in the space t, x, y) for which $t \gg 1$ and the system (5) have a solution which is naturally called the wave zone. If x/t, y/t are in the wave zone then the phase function S in (4) has stationary points and Γ_n decreases as t^{-1} for $t \to \infty$. Outside the wave zone the integral (4) has no stationary points and Γ_n decreases exponentially as $t \to \infty$.

As is mentioned in [2], the wave zone is bounded by two closed curves, the leading and trailing fronts, in every case for approximately the real distributions N(z), U(z), V(z). Let us put $\lambda = k\cos\psi$, $\mu = k\sin\psi$. Then the leading front is the curve which the point $\alpha = \partial\omega_n(\lambda, \mu)/\partial\lambda$, $\beta = \partial\omega_n(\lambda, \mu)/\partial\mu$ describes as $k \to 0$ and $0 < \psi < 2\pi$; the trailing front is the limit of the this curve as $k \to \infty$ and $0 < \psi < 2\pi$. In the absence of flows [when $\omega_n(\lambda, \mu) = \omega_n(k)$, $k^2 = \lambda^2 + \mu^2$] the leading front is the circle $\alpha^2 + \beta^2 = C_n^2$, where C_n is the maximal group velocity of the n-th mode $(C_n = \max_k \partial\omega_n/\partial k)$ and the trailing front shrinks to the origin $\alpha = \beta = 0$ since $\partial\omega_n/\partial k \to 0$ as $k \to \infty$. For the trailing front analytic expressions are found in [2]. Determination of the shape of the leading front requires numerical computations. An approximate method for computing the leading front position is proposed in this paper, which has sufficient accuracy and reduces the volume of calculations by an order.

Substituting $\lambda = k\cos\psi$, $\mu = k\sin\psi$, $\omega_n = k\xi_n(k, \psi)$ in (5), we obtain an expression for the leading front as $k \to 0$ [2]

$$\alpha = \frac{x}{t} = \xi(\psi)\cos\psi - \xi'_\psi(\psi)\sin\psi, \quad \beta = \frac{y}{t} = \xi(\psi)\sin\psi + \xi'_\psi(\psi)\cos\psi, \tag{6}$$

where $\xi(\psi) = \xi_n(\psi)$ is the n-th eigenvalue dependent on the parameter ψ for the spectral problem

$$\frac{d}{dz}\left[(\xi_n - F(z))^2 \varphi_z'\right] + N^2(z)\varphi = 0; \quad \varphi_n = 0 \quad (z = 0, -H). \tag{7}$$

Here $F(z) = U\cos\psi + V\sin\psi$ and $N(z)$ is the Brunt-Väisälä frequency. The derivative ξ_ψ' is expressed by a quadrature in terms of the eigenfunction φ_n:

$$\xi_\psi'(\psi) = -\int_{-H}^{0} (\xi - F)\left[U\sin\psi - V\cos\psi\right](\varphi_z')^2 dz \left[\int_{-H}^{0} (\xi - F)(\varphi_z')^2 dz\right]^{-1}.$$

Therefore, the numerical solution of the spectral problem (7) for all ψ must be found to compute the leading front. To reduce the volume of calculations substantially we use the perturbation method. We set $U = \varepsilon u(z)$, $V = \varepsilon v(z)$, $F(z) = \varepsilon[u\cos\psi + v\sin\psi] = \varepsilon f^*$ and we seek the solution of the spectral problem (7) in the form of the series $\xi_n(\psi) = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \ldots$, $\varphi_n(z) = \zeta_0(z) + \varepsilon\zeta_1(z, \psi) + \varepsilon^2\zeta_2(z, \psi) + \ldots$. Substituting the expressions written down into (7) we obtain

$$L^*\zeta_0 = \zeta_{0zz}'' + \frac{N^2(z)}{\eta_0^2}\zeta_0 = 0; \tag{8}$$

$$L^*\zeta_1 = -\frac{2}{\eta_0}\left[(\eta_1 - f^*)\zeta_{0z}'\right]_z'; \tag{9}$$

$$L^*\zeta_2 = -\frac{2}{\eta_0}\left[(\eta_1 - f^*)\zeta_{1z}'\right]_z' - \frac{1}{\eta_0^2}\left[(\eta_1 + 2\eta_2\eta_0 - 2f^*\eta_1 + f^{*2})\zeta_{0z}'\right]_z'. \tag{10}$$

There follows $\zeta_n(0) = \zeta_n(-H)$ from the boundary conditions in (7). However, this condition does not determine the function ζ_0, ζ_1, ζ_2 uniquely since there results from (8) that ζ_0 is defined to the accuracy of a constant factor while ζ_n is determined to the accuracy of the solution of the homogeneous equation $L^*\zeta_n = 0$, i.e., to the accuracy of the component const × $\zeta_0(z)$. To reduce this indeterminacy, we introduce the additional condition $\varphi_n'(0) = 1$. Then $\zeta_{0z}'(0) = 1$ and for $\zeta_{nz}'(0) = 0$ we obtain the boundary conditions

$$\zeta_n = 0 \quad (z = 0, -H), \; \zeta_{0z}'(0) = 1, \quad \zeta_{nz}'(0) = 0 \quad (n \geqslant 1) \tag{11}$$

for (8)-(10).

The expression (8) is the equation for the eigenfunctions φ_n in the absence of a flow, hence $\eta_0 = C_n$, $\zeta_0 = \varphi_n(z)/\varphi_{nz}'(0)$. The condition for solvability of (9) is orthogonality of its right side to the solution $\zeta_0(z)$ of the homogeneous equation. We obtain from this condition

$$\varepsilon\eta_1 = \frac{\varepsilon\int_{-H}^{0} f^*(z)(\zeta_{0z}')^2 dz}{\int_{-H}^{0}(\zeta_{0z}')^2 dz} = A\cos\psi + B\sin\psi,$$

$$A = \frac{1}{M}\int_{-H}^{0} U(\zeta_{0z}')^2 dz, \quad B = \frac{1}{M}\int_{-H}^{0} V(\zeta_{0z}')^2 dz, \quad M = \int_{-H}^{0}(\zeta_{0z}')^2 dz.$$

And according to (6) we find parametric equations for the leading front

$$C_x = \frac{x}{t} = \eta_0\cos\psi + A, \quad C_y = \frac{y}{t} = \eta_0\sin\psi + B. \tag{12}$$

Therefore, in a first approximation the leading front is a circle with radius $\eta_0 t$ (i.e., with radius $C_n t$) and center $x = At$, $y = Bt$. In other words, in the first approximation the unperturbed leading front moves as a single whole at the velocity $\mathbf{U} = (A, B)$, where A and B are the mean values of the flow velocity components $U(z)$ and $V(z)$ taken with the weight $(\zeta_{0z}')^2$. The circle 1 in Fig. 1 is the leading front for a medium without a flow, while 2 is the leading front when there is a one-dimensional shear flow in the medium and only the first approximation is taken into account in the construction.

We introduce a moving coordinate system $x' = x - At$, $y' = y - Bt$. The corrections to the first approximation for the leading front vanish at the coordinates t, x', y' and the calculations needed to construct the second approximation are simplified (we return to the original variables t, x, y in the final expression for the group velocity components). The expression for η_2 is obtained from the orthogonality condition of the right side of (10) to the solution of the homogeneous equation ζ_0:

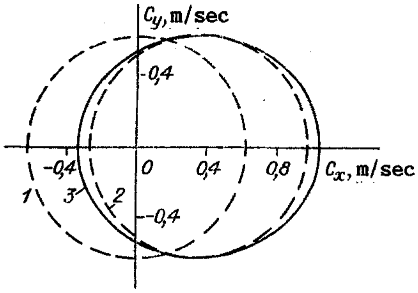

Fig. 1

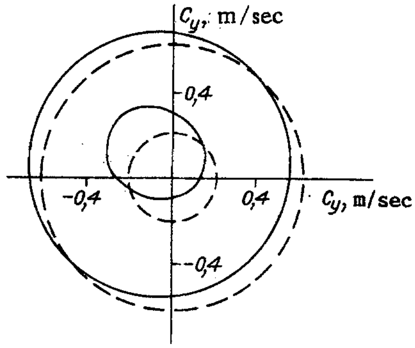

Fig. 2

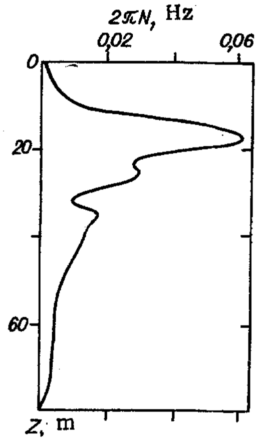

Fig. 3

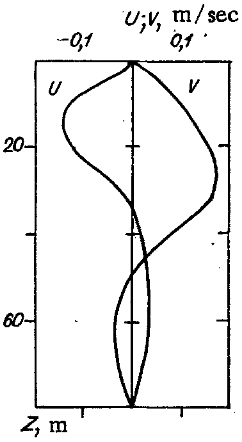

Fig. 4

$$\eta_2(\psi) = \frac{2\eta_0 \int\limits_{-H}^{0} f^* \zeta'_{1z}\zeta'_{0z} dz - \int\limits_{-H}^{0} f^{*2} (\zeta'_{0z})^2 dz}{2\eta_0 \int\limits_{-H}^{0} (\zeta'_{0z})^2 dz}.$$

Here the function ζ_1 is the solution of (9). Taking into account that $\eta_1 = 0$ and $f^* = u\cos\psi + v\sin\psi$, we represent ζ_1 in the form $\varepsilon\zeta_1 = \zeta_u \cos\psi + \zeta_v \sin\psi$, where ζ_u and ζ_v are solutions of the following equations $L^*\zeta_u = \frac{2}{\eta_0}\,[U\zeta'_{0z}]'_z$, $L^*\zeta_v = \frac{2}{\eta_0}\,[V\zeta'_{0z}]'_z$.

Explicitly extracting the dependence of η_2 on the parameter ψ and going over to the group velocities C_x, C_y, we obtain

$$\begin{aligned} C_x &= \eta_0 \cos\psi + A + [A_2^0 - A_2^{10}\cos\psi(1+\sin^2\psi) - A_2^{20}\sin^3\psi + A_2^{30}\sin^2\psi\cos\psi], \\ C_y &= \eta_0 \sin\psi + B + [B_2^0 - B_2^{10}\sin\psi(1+\cos^2\psi) - B_2^{20}\cos^3\psi + B_2^{30}\cos^2\psi\sin\psi], \end{aligned} \tag{13}$$

where

$$A_2^0 = C_1 \int\limits_{-H}^{0} \zeta'_{1z}\zeta'_{0z} U dz; \quad B_2^0 = C_1 \int\limits_{-H}^{0} \zeta'_{1z}\zeta'_{0z} V dz;$$

$$A_2^{10} = B_2^{30} = \frac{C_1}{2\eta_0} \int\limits_{-H}^{0} U^2 (\zeta'_{0z})^2 dz; \quad A_2^{20} = B_2^{20} = \frac{C_1}{\eta_0} \int\limits_{-H}^{0} UV (\zeta'_{0z})^2 dz;$$

$$A_2^{30} = B_2^{10} = \frac{C_1}{2\eta_0} \int\limits_{-H}^{0} V^2 (\zeta'_{0z})^2 dz; \quad C_1 = \left[\int\limits_{-H}^{0} (\zeta'_{0z})^2 dz\right]^{-1}.$$

Therefore, the main part of the machine time to be expended is used to find the functions ζ_0, ζ_1 and their derivatives, after which the quadrature formulas are evaluated. The second approximation (13) permits description of the wave front movement as a whole and its deformation (the curve 3 in Fig. 1 is constructed with the second approximation taken into account).

A simple method of solving the spectral problem $L^*\zeta_0 = 0$, $\zeta_0 = 0$ ($z = 0, -H$) is presented in [3]. The Brunt–Väisälä frequency in this method is approximated by a piecewise-constant function, i.e., the whole interval of values $[-H, 0]$ is divided into layers, in each of which the solution is written down in analytic form and the integration of (8) is reduced to converting the function and its derivative or the impedance $Z = \zeta_0/\zeta'_{0z}$ from one horizon to another over the whole layer. Then the equation for the eigenvalues has the form $Z_0^J - Z_{-H}^J = 0$, where Z_{-H}^J is the impedance converted from the bottom to the J-th horizon, while Z_0^J is the impedance converted from the surface to the same horizon. The number of the eigennumber is determined by the number of zeros at the appropriate eigenfunction. Integration of (9) is performed by an analogous method, the solution in each layer is expressed in terms of the solution of the homogeneous equation and the right side of (9). A program is written on the basis of this method and the "exact" and approximate methods are compared. It is shown that for real (practically for all stable) flows the relative error in determining the coordinates of the leading fronts by using the perturbation method for the first modes does not exceed 10% (this is totally adequate for the processing of full-scale data).

As an illustration of the influence of shear flows on internal waves, the leading wave fronts of the first and second internal wave modes are represented in Fig. 2 for a medium with a two-dimensional shear flow (solid curve). The distributions of the Brunt–Väisälä frequency and the flow velocity components were taken from results of measurements and are presented in Figs. 3 and 4. Corresponding fronts for media without flows are shown by dashed lines for comparison.

It is seen from Figs. 1 and 2 that the presence of flows results in a substantial change in the wave front location, and therefore, of the whole internal wave field also. These changes can be computed by using the presented sufficiently accurate and simple algorithm.

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DISPLACING OIL WITH HOT WATER AND STEAM

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UDC 532.546+622.276.65

Exact solutions are derived via the approach of [1-3] for frontal oil displacement by steam or steam–water mixtures [4] in the large-scale approximation, i.e., where we neglect capillary, diffusion, and nonequilibrium effects as well as thermal conduction in the stratum in the displacement direction. It is assumed that the water and steam when present together in the porous medium have equal mobilities. Then three-phase flows, if they occur, amount to two-phase ones, with the aqueous phase a mixture of water and steam. The thermal-wave structure is determined by the nonlinear temperature dependence for the specific heat content in the generalized water phase, which is independent of the saturation distribution. For example, if saturated steam is pumped into the stratum, the temperature alters stepwise, with the step corresponding to the steam condensing to cold water. In superheated-steam displacement, there is a two-stage temperature distribution, with a slow front in which the steam cools to the transition point and a more rapid condensation one. The relation between the displacing capacity and the specific heat content is of turning-point type: it is maximal for hot water and decreases on going to cold water and steam. Therefore, one cannot construct the solution in the large-scale approximation without considering the internal step structure corresponding to the condensation front, where the evolutionary conditions are not obeyed. The condition for a continuous internal structure is related to the diffuseness in

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